

# 1 Sequences

## 1.1 General definitions

A **sequence** is a list of numbers that is written in a defined order, ascending or descending or neither, following a specific rule.

For example:

- 2, 4, 6, 8, ...
- 1, 3, 9, 27, 81, ...
- 1000, -500, 250, -125, ...
- 1, 1, 2, 3, 5, 8, 13, 21, ...

**Can you determine the rule of these sequences?**

A sequence can be either finite or infinite.

- A **finite sequence** has a fixed number of terms.
- An **infinite sequence** has an infinite number of terms.

For example:

- 8, 6, 4, 2, 0, -2 is finite because it ends after the sixth term.
- 1, 3, 5, 7, ... is infinite because the three dots (called the "ellipsis") at the end indicate that the sequence is never ending or continues indefinitely.

A term in a sequence is named using the notation  $u_n$ , where  $n$  is the position of the term in the sequence. The first term is called  $u_1$ , the second term is  $u_2$ , etc.

For example:

- The sequence 2, 4, 6, 8, 10 has  $u_1 = 2$ ,  $u_2 = 4$ , etc.
- The sequence 1, -3, 9, -27, ... has  $u_1 = 1$ ,  $u_2 = -3$ , etc.

A formula or expression that mathematically describes the pattern of the sequence can be found for the general term,  $u_n$ .

For example:

- 1, 4, 7, 10, 13, ...  $\rightarrow u_n = 3n - 2$
- 1, 4, 9, 16, 25, ...  $\rightarrow u_n = n^2$
- 3, -6, 12, -24, 48, ...  $\rightarrow u_n = 3(-2)^{n-1}$
- 1, 1, 2, 3, 5, 8, 13, 21, ...  $\rightarrow u_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$

**How to determine the general term of a sequence?**

## 1.2 Arithmetic sequences - basic definition

A sequence is called **arithmetic** when the same value (called the "common difference") is added to each term to obtain the next term.

For example:

- 10, 15, 20, 25, 30, ... is an arithmetic sequence because 5 is added to each term.
- 10, 7, 4, 1, -2, -5, ... is an arithmetic sequence because -3 is added to each term.
- 1, 2, 4, 7, 11, 16, ... is not an arithmetic sequence because the differences between consecutive terms are not the same:  $2 - 1 \neq 4 - 2$

## 1.3 Geometric sequences - basic definition

A sequence is called **geometric** when each term is multiplied by the same value (called the "common ratio") to get the next term.

For example:

- 2, 4, 8, 16, 32, ... is a geometric sequence because each term is obtained by the previous term multiplied by 2.
- 200, 100, 50, 25, 12.5, ... is a geometric sequence because each term is obtained by the previous term multiplied by  $\frac{1}{2}$ .
- 1, 2, 6, 24, 120, ... is not a geometric sequence because the ratios between consecutive terms are not the same:  $\frac{2}{1} \neq \frac{6}{2}$

## 1.4 Recursive sequence

A **recursive sequence** uses the previous term or terms to find the next term. The general term will include the notation  $u_{n-1}$ , which means "the previous term".

For example:

- $u_n = 2u_{n-1} + 5$  and  $u_1 = -4$   
 $\rightarrow u_2 = 2(-4) + 5 = -3$   
 $u_3 = 2(-3) + 5 = -1$  , etc.
- $u_n = (u_{n-1})^2 - 3$  and  $u_1 = 2$   
 $\rightarrow u_2 = 2^2 - 3 = 1$   
 $u_3 = 1^2 - 3 = -2$  , etc.
- $u_n = u_{n-1} + u_{n-2}$  and  $u_1 = 1$  ,  $u_2 = 1$   
 $\rightarrow u_3 = 1 + 1 = 2$   
 $u_4 = 2 + 1 = 3$  , etc.

## 1.5 Series

Sequence Example	Series Example
5, 10, 15, 20, 25	5 + 10 + 15 + 20 + 25
A sequence is an ordered list of numbers following a particular pattern or rule.	A series is formed by adding the terms of a sequence.

- A **series** is created when the terms of a sequence are added together.
- A **finite series** has a fixed number of terms.
- An **infinite series** continues indefinitely.

For example,

- The series  $7 + 5 + 3 + 2 + (-1) + (-3)$  is finite because it ends after the sixth term.
- The series  $10 + 8 + 6 + 4 + \dots$  is infinite because the ellipsis indicates that the series continues indefinitely.

A series can be written in a form called **sigma notation** :  $\sum_{r=1}^n u_r$

where

- $u_r$  is the general term of the series
- $r = 1$  indicates that the series starts with the first term of the sequence
- " $n$ " is the upper limit of the series
- " $r$ " is the index and will be consecutive integers from 1 to  $n$ .

For example,

- $\sum_{r=1}^n u_r = u_1 + u_2 + u_3 + \dots + u_n$
- $\sum_{n=1}^5 3n - 2 = (3 \times 1 - 2) + (3 \times 2 - 2) + (3 \times 3 - 2) + (3 \times 4 - 2) + (3 \times 5 - 2)$
- $\sum_{n=1}^{\infty} 3 \times 2^n = 3 \times 2^1 + 3 \times 2^2 + 3 \times 2^3 + \dots$
- $\sum_{r=5}^n u_r = u_5 + u_6 + u_7 + \dots + u_n$   
(A series does not need to start at the first term!)