1 Sequences

1.1 General definitions

A **sequence** is a list of numbers that is written in a defined order, ascending or descending or neither, following a specific rule.

For example:

- 2, 4, 6, 8, ...
- 1, 3, 9, 27, 81, ...
- 1000, −500, 250, −125, ...
- 1, 1, 2, 3, 5, 8, 13, 21, ...

Can you determine the rule of these sequences?

A sequence can be either finite or infinite.

- A **finite sequence** has a fixed number of terms.
- An **infinite sequence** has an infinite number of terms.

For example:

- 8, 6, 4, 2, 0, -2 is finite because it ends after the sixth term.
- 1, 3, 5, 7, ... is infinite because the three dots (called the ″ellipsis″) at the end indicate that the sequence is never ending or continues indefinitely.

A term in a sequence is named using the notation u_n, where n is the position of the term in the sequence. The first term is called u_1, the second term is u_2, etc.

For example:

- The sequence 2, 4, 6, 8, 10 has $u_1 = 2$, $u_2 = 4$, etc.
- The sequence 1, −3, 9, −27, ... has $u_1 = 1$, $u_2 = -3$, etc.

A formula or expression that mathematically describes the pattern of the sequence can be found for the general term, u_n.

For example:

- 1, 4, 7, 10, 13, ... $\rightarrow$ $u_n = 3n - 2$
- 1, 4, 9, 16, 25, ... $\rightarrow$ $u_n = n^2$
- 3, −6, 12, −24, 48, ... $\rightarrow$ $u_n = 3(-2)^{n-1}$
- 1, 1, 2, 3, 5, 8, 13, 21, ... $\rightarrow$ $u_n = \dfrac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$

How to determine the general term of a sequence?

1.2 Arithmetic sequences - basic definition

A sequence is called **arithmetic** when the same value (called the ″common difference″) is added to each term to obtain the next term.

For example:

- 10, 15, 20, 25, 30, ... is an arithmetic sequence because 5 is added to each term.
- 10, 7, 4, 1, -2, -5, ... is an arithmetic sequence because -3 is added to each term.
- 1, 2, 4, 7, 11, 16, ... is not an arithmetic sequence because the differences between consecutive terms are not the same: $2-1 \neq 4-2$

1.3 Geometric sequences - basic definition

A sequence is called **geometric** when each term is multiplied by the same value (called the ″common ratio″) to get the next term.

For example:

- 2, 4, 8, 16, 32, ... is a geometric sequence because each term is obtained by the previous term multiplied by 2.
- 200, 100, 50, 25, 12.5, ... is a geometric sequence because each term is obtained by the previous term multiplied by $\frac{1}{2}$.
- 1, 2, 6, 24, 120, ... is not a geometric sequence because the ratios between consecutive terms are not the same: $\frac{2}{1} \neq \frac{6}{2}$

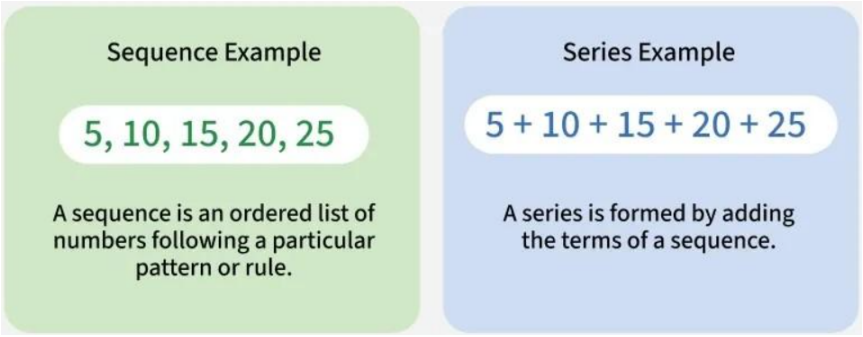
1.4 Recursive sequence

A **recursive sequence** uses the previous term or terms to find the next term. The general term will include the notation u_{n-1}, which means ″the previous term″.

For example:

- $u_n = 2u_{n-1} + 5$ and $u_1 = -4$
  $\rightarrow\ u_2 = 2(-4) + 5 = -3$
  $u_3 = 2(-3) + 5 = -1$, *etc.*
- $u_n = (u_{n-1})^2 - 3$ and $u_1 = 2$
  $\rightarrow\ u_2 = 2^2 - 3 = 1$
  $u_3 = 1^2 - 3 = -2$, *etc.*
- $u_n = u_{n-1} + u_{n-2}$ and $u_1 = 1$, $u_2 = 1$
  $\rightarrow\ u_3 = 1 + 1 = 2$
  $u_4 = 2 + 1 = 3$, *etc.*

1.5 Series



- A **series** is created when the terms of a sequence are added together.
- A **finite series** has a fixed number of terms.
- An **infinite series** continues indefinitely.

For example,

- The series $7+5+3+2+(-1)+(-3)$ is finite because it ends after the sixth term.
- The series $10+8+6+4+...$ is infinite because the ellipsis indicates that the series continues indefinitely.

A series can be written in a form called **sigma notation** : $\sum_{r=1}^{n} u_r$

where

- u_r is the general term of the series
- $r=1$ indicates that the series starts with the first term of the sequence
- $''n''$ is the upper limit of the series
- $''r''$ is the index and will be consecutive integers from 1 to n.

For example,

- $\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + ... + u_n$
- $\sum_{n=1}^{5} 3n-2 = (3\times 1-2)+(3\times 2-2)+(3\times 3-2)+(3\times 4-2)+(3\times 5-2)$
- $\sum_{n=1}^{\infty} 3\times 2^n = 3\times 2^1 + 3\times 2^2 + 3\times 2^3 + ...$
- $\sum_{r=5}^{n} u_r = u_5 + u_6 + u_7 + ... + u_n$

  (A series does not need to start at the first term!)